Trig Identities Review — Key

1. $(1+\sin x)\cdot(\sec x - \tan x) = \cos x$

$$\text{LHS} = \sec x - \tan x + \sin x \cdot \sec x - \sin x \cdot \tan x$$

$$= \frac{1}{\cos x} - \frac{\sin x}{\cos x} + \frac{\sin x}{\cos x} - \frac{\sin^2 x}{\cos x}$$

$$= \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x}$$

$$= \frac{1-\sin^2 x}{\cos x}$$

$$= \frac{\cos^2 x}{\cos x} = \cos x = \text{RHS}$$

2. $\sin x\,(1+\cot^2 x) = \csc x$

$$\text{LHS} = \sin x\left(1+\frac{\cos^2 x}{\sin^2 x}\right)$$

$$= \sin x\left(\frac{\sin^2 x + \cos^2 x}{\sin^2 x}\right)$$

$$= \sin x \cdot \frac{1}{\sin^2 x}$$

$$= \frac{\sin x}{\sin^2 x} = \frac{1}{\sin x} = \csc x = \text{RHS}.$$

3. $\csc^2 x - \cot^2 x = 1$

$$LHS = \frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}$$

$$= \frac{1 - \cos^2 x}{\sin^2 x} = \frac{\sin^2 x}{\sin^2 x} = 1 = RHS.$$

4. $\tan x \cdot \sin^2 x \cdot \cos x = \sin^3 x$

$$LHS = \frac{\sin x}{\cancel{\cos x}} \cdot \sin^2 x \cdot \cancel{\cos x}$$

$$= \sin^3 x = RHS.$$

5. $\dfrac{\cos x}{\sin x} + \dfrac{\sin x}{\cos x} = \sec x \cdot \csc x$

$$LHS = \frac{\cos^2 x + \sin^2 x}{\sin x \cdot \cos x} = \frac{1}{\sin x \cdot \cos x} = \sec x \cdot \csc x = RHS.$$

6. $\dfrac{1}{\cos x} - \cos x = \tan x \cdot \sin x$

$$LHS = \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}$$

$$= \frac{1 - \cos^2 x}{\cos x}$$

$$= \frac{\sin^2 x}{\cos x} = \frac{\sin x}{\cos x} \cdot \sin x = \tan x \cdot \sin x = RHS.$$

7. $\sec x \cdot \tan x \cdot \csc x = \sec^2 x$

$$LHS = \frac{1}{\cos x} \cdot \frac{\cancel{\sin x}}{\cos x} \cdot \frac{1}{\cancel{\sin x}}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x = RHS.$$

8. $(\cos x + \sin x)^2 = 1 + 2\cos x \cdot \sin x$

$$LHS = \cos^2 x + 2\cos x \cdot \sin x + \sin^2 x$$

$$= \cos^2 x + \sin^2 x + 2\cos x \cdot \sin x$$

$$= 1 + 2\cos x \cdot \sin x = RHS.$$

9. $\dfrac{\csc x \cdot \tan x}{\sec x} = 1$

$$LHS = \frac{\frac{1}{\cancel{\sin x}} \cdot \frac{\cancel{\sin x}}{\cos x}}{\frac{1}{\cos x}}$$

$$= \frac{\frac{1}{\cos x}}{\frac{1}{\cos x}} = 1 = RHS.$$

10. $\dfrac{1 - \cos^4 x}{1 + \cos^2 x} = \sin^2 x$

$$LHS = \frac{\cancel{(1 + \cos^2 x)}(1 - \cos^2 x)}{\cancel{1 + \cos^2 x}}$$

$$= 1 - \cos^2 x = \sin^2 x = RHS.$$

11. $\csc^4 x - \cot^4 x = \dfrac{1+\cos^2 x}{\sin^2 x}$

$$\begin{aligned}
\text{LHS} &= (\csc^2 x + \cot^2 x)(\csc^2 x - \cot^2 x) \\
&= \left(\frac{1}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x}\right)\left(\frac{1}{\sin^2 x} - \frac{\cos^2 x}{\sin^2 x}\right) \\
&= \left(\frac{1+\cos^2 x}{\sin^2 x}\right)\left(\frac{1-\cos^2 x}{\sin^2 x}\right) \\
&= \left(\frac{1+\cos^2 x}{\sin^2 x}\right)\left(\cancel{\frac{\sin^2 x}{\sin^2 x}}\right) \\
&= \frac{1+\cos^2 x}{\sin^2 x} = \text{RHS.}
\end{aligned}$$

12. $(\tan x + \cot x)^2 = \sec^2 x + \csc^2 x$

$$\begin{aligned}
\text{LHS} &= \tan^2 x + 2\tan x \cdot \cot x + \cot^2 x \\
&= \frac{\sin^2 x}{\cos^2 x} + 2\,\frac{\cancel{\sin x}}{\cancel{\cos x}} \cdot \frac{\cancel{\cos x}}{\cancel{\sin x}} + \frac{\cos^2 x}{\sin^2 x} \\
&= \frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\sin^2 x} + 2 \\
&= \tan^2 x + \cot^2 x + 2 \\
&= (\sec^2 x - 1) + (\csc^2 x - 1) + 2 \\
&= \sec^2 x + \csc^2 x = \text{RHS.}
\end{aligned}$$

13. $\dfrac{1+\sin x}{1-\sin x} - \dfrac{1-\sin x}{1+\sin x} = 4\tan x \cdot \sec x$

$$LHS = \frac{(1+\sin x)^2 - (1-\sin x)^2}{1-\sin^2 x}$$

$$= \frac{1+2\sin x + \sin^2 x - (1 - 2\sin x + \sin^2 x)}{1-\sin^2 x}.$$

$$= \frac{4\sin x}{\cos^2 x} = 4 \cdot \tan x \cdot \sec x = RHS.$$

14. $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$

$$LHS = \cos\frac{\pi}{2} \cdot \cos x - \sin\frac{\pi}{2} \sin x$$

$$= 0 \cdot \cos x - 1 \cdot \sin x$$

$$= -\sin x = RHS.$$

15. $2\cot x \cdot \sin x = \csc x \cdot \sin(2x)$

$LHS = 2\dfrac{\cos x}{\sin x} \cdot \sin x$ $\qquad$ $RHS = \dfrac{1}{\sin x} \cdot 2\sin x \cdot \cos x$

$= 2\cos x$ $\qquad$ $= 2\cos x$

$LHS = RHS.$

16. $\cos(\pi + x) = -\cos x$

$$\text{LHS} = \cos\pi \cdot \cos x - \sin\pi \cdot \sin x$$

$$= -1 \cdot \cos x - 0 \cdot \sin x$$

$$= -\cos x = \text{RHS}.$$

17. $\tan x + \cot x = 2\csc(2x)$

$$\text{LHS} = \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} \qquad \text{RHS} = \frac{2}{\sin(2x)}$$

$$= \frac{\sin^2 x + \cos^2 x}{\sin x \cdot \cos x} \qquad = \frac{2}{2\sin x \cdot \cos x}$$

$$= \frac{1}{\sin x \cdot \cos x} \qquad = \frac{1}{\sin x \cdot \cos x}$$

$$\text{LHS} = \text{RHS}.$$

18. $\sin(2x) \cdot \cos x - \cos(2x) \cdot \sin x = \sin x$

$$\text{LHS} = 2\sin x \cdot \cos^2 x - (\cos^2 x - \sin^2 x) \cdot \sin x$$

$$= 2\sin x \cdot \cos^2 x - \sin x \cdot \cos^2 x + \sin^3 x$$

$$= \sin x \cdot \cos^2 x + \sin^3 x$$

$$= \sin x(\cos^2 x + \sin^2 x)$$

$$= \sin x = \text{RHS}.$$

19. $\frac{\sin(2x)}{2} \cdot \cos x = \sin x - \sin^3 x$

$$\text{LHS} = \frac{2\sin x \cdot \cos^2 x}{2}$$

$$= \sin x \cdot \cos^2 x$$

$$= \sin x\,(1 - \sin^2 x)$$

$$= \sin x - \sin^3 x = \text{RHS.}$$

20. $\sec^2 x \cdot \csc^2 x = \frac{4}{\sin^2(2x)}$

$$\text{LHS} = \frac{1}{\cos^2 x} \cdot \frac{1}{\sin^2 x}$$

$$= \frac{1}{\sin^2 x \cdot \cos^2 x}$$

$$\text{RHS} = \frac{4}{(2\sin x \cos x)^2}$$

$$= \frac{4}{4\sin^2 x \cdot \cos^2 x}$$

$$= \frac{1}{\sin^2 x \cdot \cos^2 x}$$

LHS = RHS.

PART II

1. $\cos 75 = \cos(30 + 45)$

$= \cos 30 \cos 45 - \sin 30 \sin 45$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

$\cos 75 = \frac{\sqrt{6} - \sqrt{2}}{4}$

2. $\sin 15 = \sin(45 - 30)$

$= \sin 45 \cos 30 - \sin 30 \cos 45$

$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \frac{\sqrt{2}}{2}$

$\sin 15 = \frac{\sqrt{6} - \sqrt{2}}{4}$

3. $\sin \frac{7\pi}{12} = \sin\left(\frac{3\pi}{12} + \frac{4\pi}{12}\right)$

$= \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right)$

$= \sin \frac{\pi}{4} \cdot \cos \frac{\pi}{3} + \sin \frac{\pi}{3} \cdot \cos \frac{\pi}{4}$

$= \frac{\sqrt{2}}{2} \cdot \frac{1}{2} + \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{2}}{2}$

$\sin \frac{7\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}$

4. $\cos(-120) = \cos(60^\circ - 180^\circ)$

$$= \cos 60^\circ \cdot \cos 180^\circ + \sin 60^\circ \cdot \sin 180^\circ$$

$$= \left(\tfrac{1}{2}\right)(-1) + \left(\tfrac{\sqrt{3}}{2}\right)(0)$$

$$\cos(-120) = -\frac{1}{2}$$

5. $\tan \frac{13\pi}{12} = \tan\left(\frac{9\pi}{12} + \frac{4\pi}{12}\right)$

$$= \tan\left(\frac{3\pi}{4} + \frac{\pi}{3}\right)$$

$$= \frac{\tan \frac{3\pi}{4} + \tan \frac{\pi}{3}}{1 - \tan \frac{3\pi}{4} \cdot \tan \frac{\pi}{3}}$$

$$= \frac{-1 + \sqrt{3}}{1 - (-1)\sqrt{3}}$$

$$= \frac{-1 + \sqrt{3}}{1 + \sqrt{3}} \cdot \left(\frac{1 - \sqrt{3}}{1 - \sqrt{3}}\right)$$

$$= \frac{-1 + 2\sqrt{3} - 3}{1 - 3}$$

$$= \frac{2\sqrt{3} - 4}{-2}$$

$$\tan \frac{13\pi}{12} = 2 - \sqrt{3}$$

PART III

$$\cos\alpha = \pm\sqrt{1-\sin^2\alpha} \qquad \sin\beta = \pm\sqrt{1-\cos^2\beta}$$

$$= \pm\sqrt{1-\left(\frac{4}{5}\right)^2} \qquad = \pm\sqrt{1-\left(\frac{12}{13}\right)^2}$$

$$\cos\alpha = \pm\frac{3}{5} \qquad = \pm\frac{5}{13}$$

$$\cos\alpha = \frac{3}{5} \qquad \sin\beta = \frac{5}{13}$$

1. $\sin(\alpha+\beta) = \sin\alpha \cdot \cos\beta + \sin\beta \cdot \cos\alpha$

$$= \frac{4}{5}\cdot\frac{12}{13} + \frac{5}{13}\cdot\frac{3}{5}$$

$$\sin(\alpha+\beta) = \frac{63}{65}$$

2. $\cos(\alpha+\beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$

$$= \left(\frac{3}{5}\right)\left(\frac{12}{13}\right) - \left(\frac{4}{5}\right)\left(\frac{5}{13}\right)$$

$$\cos(\alpha+\beta) = \frac{16}{65}$$

3. $\tan(\alpha+\beta) = \dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$$= \frac{\frac{63}{65}}{\frac{16}{65}} = \frac{63}{16}$$

$$\sin\alpha = \pm\sqrt{1-\cos^2\alpha} \qquad \sin\beta = \pm\sqrt{1-\cos^2\beta}$$

$$= \pm\sqrt{1-\left(\frac{5}{13}\right)^2} \qquad = \pm\sqrt{1-\left(\frac{4}{5}\right)^2}$$

$$= \pm\frac{12}{13} \qquad = \pm\frac{3}{5}$$

$$\sin\alpha = \frac{12}{13} \qquad \sin\beta = \frac{3}{5}$$

4. $$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha$$

$$= \frac{12}{13}\cdot\frac{4}{5} - \frac{3}{5}\cdot\frac{5}{13}$$

$$\sin(\alpha-\beta) = \frac{33}{65}$$

5. $$\cos(\alpha-\beta) = \cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta$$

$$= \frac{5}{13}\cdot\frac{4}{5} + \frac{12}{13}\cdot\frac{3}{5}$$

$$\cos(\alpha-\beta) = \frac{56}{65}$$

6. $$\tan(\alpha-\beta) = \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}$$

$$= \frac{\frac{33}{65}}{\frac{56}{65}} = \frac{33}{56}$$

PART IV

1. $\sin\theta + \sin 2\theta = 0$

$\sin\theta + 2\sin\theta\cos\theta = 0$

$\sin\theta(1 + 2\cos\theta) = 0$

$\sin\theta = 0$ or $1 + 2\cos\theta = 0$ $\quad\cos\theta = \frac{-1}{2}$

$\theta = 0, 2\pi$ $\quad\theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

$\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi.$

2. $\sin\theta = -\cos(2\theta)$

$= -(\cos^2\theta - \sin^2\theta)$

$\sin\theta = \sin^2\theta - \cos^2\theta$

$= \sin^2\theta - (1 - \sin^2\theta)$

$\sin\theta = 2\sin^2\theta - 1$

$2\sin^2\theta - \sin\theta - 1 = 0$

$(2\sin\theta + 1)(\sin\theta - 1) = 0$

$\sin\theta = \frac{-1}{2}$ or $\sin\theta = 1$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ $\quad\theta = \frac{\pi}{2}$

$\theta = \frac{\pi}{2}, \frac{7\pi}{6}, \frac{11\pi}{6}.$

3. $\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = -\sin\theta$

$\cos\frac{\pi}{2} \cdot \cos\frac{\pi}{3} - \sin\frac{\pi}{2} \cdot \sin\frac{\pi}{3} = -\sin\theta$

$\bar{0} \cdot \cos\frac{\pi}{3} - 1 \cdot \frac{\sqrt{3}}{2} = -\sin\theta$

$-\frac{\sqrt{3}}{2} = -\sin\theta$

$\sin\theta = \frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}$